THE CHINESE UNIVERSITY OF HONG KONG
MATH4010 Suggested solutions to homework 2

If you find any mistakes or typos, please report them to ypyang@math.cuhk.edu.hk

5.2. Solution. For any $x = \sum_{k=1}^{\infty} x_k e_k, y = \sum_{k=1}^{\infty} y_k e_k \in X$, $\forall a, b \in \mathbb{R}$, we have

$$ax + by = a\sum_{k=1}^{\infty} x_k e_k + b\sum_{k=1}^{\infty} y_k e_k = \sum_{k=1}^{\infty}(ax_k + by_k)e_k$$

and therefore

$$f(ax + by) = ax_k + by_k = af(x) + bf(y).$$

We conclude that $f : x \mapsto x_k$ is linear for every $k \in \mathbb{N}$.

5.9. Solution. For any $f \in X^*, f \neq 0$ we have $|f(x)| \leq \|f\|\|x\|$ and hence

$$\sup\left\{\frac{|f(x)|}{\|f\|} : f \in X^*, f \neq 0\right\} \leq \|x\|.$$

On the other hand, there exists a bounded linear functional f on X such that

$$\|\tilde{f}\| = 1, \quad \tilde{f}(x) = \|x\|.$$

Therefore,

$$\sup\left\{\frac{|f(x)|}{\|f\|} : f \in X^*, f \neq 0\right\} \geq \frac{|\tilde{f}(x)|}{\|\tilde{f}\|} = \frac{\|x\|}{1} = \|x\|$$

and consequently $\sup\left\{\frac{|f(x)|}{\|f\|} : f \in X^*, f \neq 0\right\} = \|x\|$.

5.30. Solution.

(a) It's clear that δ is linear. $|\delta(x)| = |x(0)| \leq \sum_{0\leq x\leq 1} |x(t)| = \|x\|$. Therefore, δ is bounded and $\|\delta\| \leq 1$.

Take $x(t) \equiv 1$ and we have

$$|\delta(x)| = 1 \leq \|\delta\|\|x\| = \|\delta\| \Longrightarrow \|\delta\| \geq 1.$$

We conclude that $\|\delta\| = 1$.

(b) Take $x_n(t) = \begin{cases} 2n - 2n^2 t, & 0 \leq t \leq \frac{1}{n} \\ 0, & \frac{1}{n} < t \leq 1 \end{cases}$. Then we have $\delta(x) = x(0) = 2n$ and

$$\|x\| = \int_0^1 |x(t)\,dt = \int_0^{\frac{1}{n}} (2n - 2n^2 t)\,dt = 1.$$

Therefore, $|\delta(x)| = 2n \leq \|\delta\|\|x\| = \|\delta\| \Longrightarrow \|\delta\| \geq 2n$ and thus δ is unbounded.